Linear analysis of a finite length plasma-filled backward wave oscillator

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Absolute instability in a plasma-filled backward wave oscillator with sinusoidally corrugated slow wave structure driven by an intense relativistic electron beam has been analyzed numerically. The maximum spatial growth rate of the plasma-filled waveguide is found to be larger than that of an optimally designed vacuum structure. The excitation of a finite length structure is investigated. Although the spatial growth rates obtained in the linear theory increase with plasma density in the finite length plasma-filled structure for the fundamental TM_{01} mode of oscillation, negative global spatial growth rates prevent the coherent oscillation for intermediate ranges of plasma density, beam current, and even for structure length.

I. INTRODUCTION

In recent years, extensive efforts have been devoted to finding microwave sources at frequency ranges and power levels beyond the capabilities of the existing ones. In the last decade, developments in gyrotrons and free-electron lasers have made it possible to produce short wavelength radiation at modest power levels. In addition to the innovation of such wholly novel classes of microwave sources, researchers are also working to improve the performances of more familiar devices such as magnetrons and backward wave oscillators (BWO's) driven by intense relativistic electron beams (IREB's).

BWO's have a long history as efficient generators of microwave power. During the past two decades, impressive results have been obtained both experimentally and theoretically with these devices. The use of IREB's as the source of energy for the microwaves has resulted in a tremendous increase in the output power from BWO's. The multiwave Cerenkov generator and relativistic diffraction generator, which have a close relationship to BWO's, attained output power levels greater than 1 GW at frequencies of tens of GHz with high efficiencies. Previously, improvements in efficiency were observed when BWO's were operated in the presence of background gases. An eightfold enhancement in output power was achieved by the injection of plasma into a BWO from an external plasma gun. Lin and Chen reported numerical simulations of the experiment, although they assumed an artificial periodic boundary condition in the axial direction which did not correspond to the experiment. They reported that a parametric beam–plasma wave interaction could not be the dominant mechanism for the efficiency enhancement and that the enhancement could be attributed to a decrease in the phase velocity of the most unstable beam mode in the presence of a background plasma.

In order to understand the elementary physics on what was happening in the plasma-filled BWO's within the scope of linear treatment, we have here developed and analyzed numerically the absolute instability in the plasma-filled BWO's. We first consider an infinitely long plasma-filled corrugated waveguide, and next proceed to finite length case which is somewhat more complicated than the former case. Analyses are made within the general frameworks of absolute instability established in the textbook of Briggs for the case of infinitely long structure, and in that of Lifshitz and Pitaevskii for finite length case. Even in the case of infinitely long structure, we assume a localized source of signals as the original perturbation. It is known that the absolute instability with this type of perturbation grows up with the particular values of complex wave number $k$ and complex angular frequency $\omega$, at a saddle point (pinch point) of a dispersion relation in which an arbitrary set of $k$ and $\omega$ can be conceived in general. To find the saddle points, we must solve the dispersion relation assuming $k$ and $\omega$ both complex. On the other hand, if one assumes an infinitely extending sinusoidal origin of perturbation in space, then an arbitrary real wave number $k$ can be assumed in the analysis of the absolute instability. We do not consider, however, such a case in our analysis. This paper presents what we believe to be the first pinch point analysis of the dispersion relation of a plasma-filled BWO. It will be shown that, according to our pinch point analysis, increased spatial growth rates in the plasma BWO's can be expected for both infinitely long and finite length structures. Of course, in general, nonlinear analyses are required to predict enhanced radiation intensities.

The organization of this paper is as follows. In Sec. II, we formulate the linear theory of an infinitely long plasma-filled BWO. The spatial and temporal growth rates of absolute instabilities for a waveguide with large corrugation depth are calculated numerically over a wide range of plasma densities. The effects of the reflection of waves assuming a special case of mode changes at both ends of a finite length slow wave structure are analyzed in Sec. III. Finally, discussion and conclusions of this work are given in Sec. IV.
II. ENHANCED SPATIAL GROWTH RATES IN AN INFINITELY LONG PLASMA-FILLED BWO

A. Formulation of the problem

We consider first an infinitely long axisymmetric cylindrical waveguide, in which inner wall radius varies, according to \( R(z) = R_0 + h \cos(kz) \), i.e., a slow wave structure. Here \( k_0 = 2\pi/z_0 \) and \( R_0, h, z_0 \) are, respectively, the mean waveguide radius, the amplitude of corrugation, and the axial length of periodicity in corrugation. The waveguide is filled completely with a uniform, cold, collisionless, and neutralized plasma of density \( N_p \). A monoenergetic relativistic electron beam of radius \( R_s < R_0 - h \), and density \( N_b \) uniform over the cross section, is moving along the axis of the waveguide with a velocity \( v \) relative to the resting background plasma. Assuming that the entire system is immersed in an infinitely strong axial guiding magnetic field, the dispersion relation, \( D(k,\omega) = 0 \), of this beam–plasma system for TM\(_{n}\) modes with field components \( E_z, E_r \), and \( B_z \) has been derived previously. As is shown in Eq. (2), the dispersion relation, \( D(k,\omega) = 0 \), of this beam–plasma system for TM\(_{n}\) modes with field components \( E_z, E_r \), and \( B_z \) has derived previously.20,21 Here, \( D \) is the value of the determinant of a square matrix with elements

\[
D_{m n} = \left[ 1 + (n - m)Q_n \right] \left( K_n C_{m n}^{+} + L_n C_{m n}^{-} \right),
\]

\[
C_{m n}^{+} = \sum_{q=0}^{\infty} \frac{(Y_n^q)^{2q + |n - m|} J_0(Y_n^q)}{2^{2q + |n - m|} (q + |n - m|)!},
\]

where \( K_n = Y_n J_0(\delta) N_0(\delta) - X_n J_1(\delta) N_1(\delta), \)

\( L_n = X_n J_0(\delta) J_1(\delta) - Y_n J_1(\delta) J_1(\delta), \)

\( X_n^2 = R_0^2 \left( \frac{\omega^2}{c^2} - k_n^2 \right) \left( 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_i^2}{\omega^2} - \frac{\omega_k^2}{\omega^2} - \frac{\omega_s^2}{\omega^2} \right), \)

\( Y_n^2 = R_0^2 \left( \frac{\omega_p^2}{c^2} - k_n^2 \right) \left( 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_i^2}{\omega^2} - \frac{\omega_k^2}{\omega^2} - \frac{\omega_s^2}{\omega^2} \right), \)

\( Q_n = \frac{k_0 k_0}{(\omega^2/c^2) - k_n^2}, \)

\( \alpha = \frac{h}{R_0}, \)

\( \delta = R_s/R_0, \)

\( k_n = k + nk_0, \)

A similar expression for \( C_{m n}^{-} \) holds, with \( J_0 \) replaced by \( N_0 \) in \( C_{m n}^{+} \) given by Eq. (2).

B. Numerical solutions to the dispersion relation

In general, there exist an infinite number of roots for high frequencies in our dispersion relation. We confine our analysis to the case of the low-frequency fundamental TM\(_{01}\) mode with \( \omega > \omega_c \). As was expected from previous works,9,10,20 four independent periodic roots of TM\(_{01}\) mode with a periodicity of real wave number \( k_0 \) are found in our dispersion relation, \( D(k,\omega) = 0 \), in which a complex \( k = k_0 + ik_r \) is obtained for a given complex \( \omega = \omega_0 + i\omega_i \), as will be discussed later in detail, the four roots are, respectively, slow and fast beam space charge waves, forward and backward structure waves. When we depict the locations of the roots of the dispersion relation in complex \( k \) plane, we must judge ourselves whether each root corresponds to growing or decaying (stable) wave. This subtle situation arises from the fact that the principal sheet of Riemann surface of multivalued functions, for example, \( \pm (X_n^{1/2})^2, \) or \( \pm (Y_n^{1/2})^2 \), included in the functional equation \( D(k,\omega) = 0 \), is automatically chosen in the process of computer calculation without any physical consideration. So, the physical interpretation that the particular root corresponds to growing or damping wave is inevitable. For that purpose, well-known Briggs–Bers criterion18 is applied. In order to accomplish their criterion in practice, the path of integration in the inverse Laplace transformation in the complex \( \omega \) plane is deformed downward to the real axis and observe the behaviors of the corresponding roots in the complex \( k \) plane. Physical meaning of this deformation corresponds to the change from the initial value problem (large positive \( \omega \)) to the steady-state problem (\( \omega = 0 \)). According to their criterion, if \( k \) crosses the real \( k \) axis during this process, then the wave is growing, i.e., convectively unstable, otherwise, it is decaying or stable. On the other hand, if the deformation of the path of integration on the \( \omega \) plane toward the real axis is prevented by merging on the complex \( k \) plane (saddle point) of two moving roots coming from the upper and lower half-planes at some positive \( \omega \), then the instability is absolute which corresponds to a BWO. The asymptotic time response of the absolute instability for a localized source of perturbation arises mainly from the contribution near this saddle point and not from the other parts of the integral along the real \( \omega \) axis.18 In general, \( \omega \) and \( k \) at the saddle point are both complex14,18,19 \((k = k_0 + ik_r, \omega = \omega_0 + i\omega_i)\), and the absolute instability in an infinitely long structure grows up asymptotically with temporal and spatial factor \( t^{-1/2} \exp [i(k_z z - \omega_z t)] \), as is shown in Eq. (2.22) in Ref. 18 or equivalently in Eq. (6.212) in Ref. 19.

In the numerical analysis presented here, the following parameters are adopted unless specified otherwise; the relativistic factor \( \gamma = 2.23, R_0 = 0.9 \text{ cm}, R_s = 1.445 \text{ cm}, h = 0.445 \text{ cm}, z_0 = 1.67 \text{ cm}, \) and \( N_b = 2.09 \times 10^{11} \text{ cm}^{-3} \) which corresponds to the beam current \( I_b = 2.3 \text{ kA}. \)13,14 The background plasma density \( N_p \) is considered to be a variable. For the assumed parameters, we must calculate the dispersion equation, \( D(k,\omega) = 0 \), which is here truncated to \( 9 \times 9 \) determinant composed of each term up to the order of \( \alpha^{10} \) in Eq. (2), to obtain the numerical accuracy in periodicity within 1% for two periods of real wave number \( k_0 = 3.7624 \text{ cm}^{-1} \).20

The method of numerical computation devised by one of the authors (T.W.) (Ref. 22) is as follows: Since the Taylor expansion of \( D(k,\omega) = 0 \) around the root gives an equation of a circle, the solution of \( k \) for a given complex \( \omega \) can be found at the center of the contour circles of \(|D| \) in the complex \( k \) plane. An example of the contour mapping devised for the present purpose is shown in Fig. 1, where \( N_c = 2 \times 10^{12} \text{ cm}^{-3}, N_b = 5.15 \times 10^{10} \text{ cm}^{-3}, \) and \( \omega/2\pi = (14.65 + i 0.04) \text{ GHz} \). The size of the white heart-shaped part

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shown by IB (an iceberg floating on the sea) located around \(\omega = k_n v\) near the real \(k\) axis increases with \(N_b\). In IB, \(|D|\) is extremely large of the order of \(10^{30}\) or more. In other parts in Fig. 1, the values of \(|D|\) are much smaller than those in IB. IB is comparatively immobile and the observed four roots move around it quickly, when \(\omega_i\) is decreased toward zero from a large positive value keeping \(\omega_r\) constant. The roots corresponding to decaying and growing waves are denoted, respectively, by \(E\) and \(G\), which are judged from the Briggs–Bers criterion. These four roots appear in the complex \(k\) plane with a period of real wave number \(k_{\omega_r}\) as is expected from Floquet theorem.

Taking the roots in Fig. 1 as a key to the initial data for our analysis, a Newton iteration method\textsuperscript{22} for complex \(\omega\) and \(k\) is devised to find successive changes in the loci of the roots of the dispersion relation, when \(\omega_i\) is changed gradually. An example of the variation of the loci of the roots in the complex \(k\) plane is shown in Fig. 2. Thin solid and dashed lines are, respectively, the constant \(\omega_r/2\pi\) and \(\omega_i/2\pi\) lines. Arrows on the solid lines show the decreasing direction of \(\omega_i\)'s toward \(\omega_i = 0\) which are marked by open circles. The roots which cross the real \(k\) axis for some \(\omega_i > 0\) correspond to the spatially growing waves. The diamond in the figure indicates the saddle point which corresponds to the absolute instability in the infinitely long plasma BWO. The thick solid line passing through the saddle point with constant real frequency, \(\omega_r/2\pi\), separates the growing roots from decaying roots near the saddle point, and the thick solid line is called here the separatrix. The definition of the separatrix is as follows. The roots of our dispersion relation are divided into two classes: The first is the waves carrying energy in positive \(z\) direction (positive group velocity), and the second is those carrying energy in negative \(z\) direction (negative group velocity). The first and the second classes are located, respectively, above and below the separatrix. This situation comes from the inverse Fourier transformation in the complex \(k\) plane. In the first (second) class, integration along an infinitely large semicircle must be executed to close the Fourier contour in the upper-half (lower-half) \(k\) plane.
where the roots \( k \) are originally located for large positive \( \omega \).

The roots marked by \( a, a', a'', b, b' \) will be discussed later in Secs. III and IV.

In Fig. 3, the loci of the roots \( k \) of the dispersion relation in the complex \( k \) plane for real \( \omega /2\pi \) (depicted on the curves) in the vicinity of beam-structure resonance are shown. Figures 3(a), 3(b), and 3(c) are, respectively, for \( N_p = 0, 1.5 \times 10^{12} \text{ cm}^{-3} \), and \( 1.7 \times 10^{12} \text{ cm}^{-3} \). The curves in the figures are periodic in the horizontal axis with periodicity of \( k_o \), and they are symmetric regarding the real \( k \) axis. The regions where one finds the complex \( k \) roots of the dispersion equation for a given complex \( \omega \) with \( \omega_r > 0 \) and \( \omega_i < 0 \) are marked, respectively, by \( \Theta \) and \( \Theta \). The absolute instability can exist only when the saddle point is located in \( \Theta \) regions. The figures depict that the background plasma strongly deforms the curves near the real \( k \) axis, and pushes them to the right with increase in \( N_p \). The saddle points in the lower half of the complex \( k \) plane are marked by the diamonds. In the cases of vacuum or low \( N_p \), the saddle point lies in the region of \( \omega_r > 0 \) and \( k_r < 0 \) as shown in Figs. 3(a) and 3(b). The instability is, therefore, absolute. The instability, however, becomes convective as \( \omega_i \) of the saddle point switches sign, when \( N_p \) is exceedingly large. This case of \( N_p = 1.7 \times 10^{12} \text{ cm}^{-3} \) is shown in Fig. 3(c) which corresponds to a plasma traveling wave tube (TWT). Here, the saddle point obtained mathematically has no meaning in physical point of view. The convective instability in this case cannot cause any coherent radiation unless an external positive feedback mechanism is devised.

The spatial growth rate, \( k_o \), and the real part, \( k_r \), at the saddle point versus the plasma density, \( N_p \), are plotted in Fig. 4(a), respectively, by solid and dashed lines. As is shown in Fig. 4(a), both \( k_o \) and \( k_r \) increase with \( N_p \). The temporal growth rate divided by \( 2\pi, \omega_o/2\pi \), and the oscillation frequency, \( \omega_o/2\pi \), at the saddle point versus \( N_p \) are also shown, respectively, by solid and dashed lines in Fig. 4(b). The temporal growth rate of the absolute instability, \( \omega_o/2\pi \), decreases with \( N_p \), and switches sign at \( N_p = 1.67 \times 10^{12} \text{ cm}^{-3} \). The instability becomes convective above that value of \( N_p \), and the system will work as a plasma TWT, because the wave is still growing spatially. The double open circle shown by \( P \) will be discussed later, when a vacuum BWO is analyzed in Fig. 5.

Next, we look for an optimal system parameter for increasing the spatial growth rate of a vacuum BWO \((N_p = 0)\). For an infinitely long waveguide, the spatial growth rate at the saddle point is found to increase with the increase in the axial length of periodicity in corrugation, \( z_0 \). For higher values in \( z_0 \), however, the absolute instability of TM0 mode is suppressed and only the convective instability remains. The numerical solution of spatial growth rate at the saddle point in the case of vacuum BWO is shown in Fig. 5 for a wide range of \( z_0 \). Other parameters are the same as those in the plasma BWO shown in Fig. 4. The solid circles on the curves indicate the growth rates corresponding to \( z_0 = 1.67 \text{ cm} \), the experimental value.\(^{13,14}\) It must be noted that the maximum spatial growth rate shown by the double open circle \( Q \) in Fig. 5 of an optimally designed vacuum BWO is considerably smaller than that shown by the double open circle.
circle $P$ of the plasma BWO in Fig. 4(a); this fact suggests that the plasma BWO (Refs. 13 and 14) may be more advantageous than an optimally designed vacuum BWO. The physical explanation of this result obtained in the present analysis is left to be tried in the future.

III. EXCITATION OF A FINITE LENGTH PLASMA-FILLED BWO

So far, we have ignored the end effects of the slow wave structure with finite length $L$. The analysis of the dispersion relation, $D(k, \omega) = 0$, presented in Sec. II under the assumption $L$ infinity may be extended to the case of finite $L$ structures, provided that $L$ is considerably larger than the average waveguide radius, $R_p$. The analysis is made in the general framework of the global instability of finite length system shown in the textbook of Lifshitz and Pitaevskii. In other words, we have considered a quasi-steady-state case, where the temporal growth of a wave is considerably smaller than its spatial growth. For systems that are limited in the axial direction, the oscillation is affected by the reflection from both ends. The oscillation in this case should be considered simultaneously as the superposition of an infinite number of waves given by $D(k, \omega) = 0$, i.e., periodic four roots all having identical complex frequency, as is shown in an example in Fig. 1. We have found that, for a given complex $\omega$ with large positive $\omega$, the three of the four roots (the two beam space charge waves and the forward structure wave) of our $D(k, \omega) = 0$ are located above the separatrix in the complex $k$ plane and having positive group velocities, while a single root (the backward structure wave) is found below the separatrix, where the group velocity is in negative direction, as was explained in the previous section. It is noted here that only one ($k, <0$) of the two beam space charge waves with positive group velocities is the source of oscillation that is spatially growing among the four roots, all the rest three roots are either decaying or stable in nature. For a sufficiently long waveguide, we may be able to assume that the spatially growing beam space charge wave, $k_+$, is only the propagating wave in the positive $z$ direction, provided $|\text{Im}(k_+)L| > 1$. It should be noted that the wave $k_+$ carries the Poynting flux in negative direction. After reflection from the right end $z = L$, the counterwave, $k_-$, which propagates in the negative $z$ direction toward $z = 0$, can create a positive feedback in the structure. Because there is only one root, the backward structure wave, $k_-$, that propagates in the negative $z$ direction, the oscillation in a finite length system will be determined by $k_+$ and $k_-$. As is shown in Fig. 6, $k_+$ (growing wave) and $k_-$ (decaying wave) locate, respectively, above and below the separatrix.

The finite length of the waveguide imposes a new constraint to the choice of a pair of roots, $k_+$ and $k_-$. Namely, in the limit of the quasi-steady-state assumption, the condition of coherency that an electromagnetic field at any given point $z$ must be unique imposes the choice of a pair of the roots $k_+$ and $k_-$ proposed by the following relation given in Eq. (65.4) of Ref. 19: $R_1, R_2; \times \exp\{i[k_+ (\omega) - k_- (\omega)]L\} = 1$, where $R_1$ and $R_2$ are, respectively, the reflection coefficients at $z = 0$ and $z = L$. This is rewritten as follows:

$$\epsilon = k_+ - k_- = -(2\pi N/L) - i[\text{Im}(R)/L], \quad (3')$$

where $R = [R_1, R_2]$ and $N = 0, \pm 1, \pm 2, \ldots$. The arbitrary integer $N$ comes from the fact that an infinite number of sheets of the Riemann surface exist for the complex logarithm function, $\text{Im}(z) = \ln|z| + i\arg(z) + 2\pi N$, namely, $\exp(2\pi N) = 1$. There is a possibility that the structure with length $L$ constitutes a cavity resonator, if $|N|$ and $R$ are comparatively large. For example, if $R = 1$ and $\text{Re}(k_+) = -\text{Re}(k_-), |N|$ becomes the number of half-waves set on the resonator length $L$. However, we do not consider such a case of a pair of $k_+$ and $k_-$ with $\text{Re}(k_+) = -\text{Re}(k_-)$ in the present analysis. To facilitate the interpretation of Eq. (3'), it may be convenient to imagine Bohr’s quantizing condition of atomic states, because they are somewhat analogous in the sense that they require a single valued physical quantity for a given spatial point and a given time $t$. Equation (3) must be satisfied for having coherent oscillation from a finite $L$ structure in the quasi-steady-state assumption limit. Unless Eq. (3) is satisfied, oscillation then stays at incoherent, noisy, and low level, no matter how each propagating wave has positive temporal growth rate. In this paper, we define this case as the stopping of oscillation.

The oscillation in a finite length slow wave structure no longer occurs at $\omega$, and $k$, of the saddle point, but at $\omega$ and $k_+$ near the saddle point and are given by Eq. (3) along with the following two equations:

$$D(k_+, \omega) = 0, \quad (4)$$

$$D(k_+ + \epsilon, \omega) = 0. \quad (5)$$

On separating the real and the imaginary parts of $\epsilon$ in Eq. (3), we have

$$\text{Re}(\epsilon) = \text{Re}(k_-) - \text{Re}(k_+) = -2\pi N/L, \quad (6)$$

$$\text{Im}(\epsilon) = \text{Im}(k_-) - \text{Im}(k_+) = -\text{Im}(R)/L > 0. \quad (7)$$

where $0 < R < 1$ has been assumed. Equation (7) holds only if $|\text{Im}(k_-)| > |\text{Im}(k_+)|$, because both $k_+$ and $k_-$ lie in the lower-half $k$ plane, as was shown in Fig. 2. To satisfy Eq. (7), therefore, the global spatial growth rate, defined by $|\text{Im}(k_+)| - |\text{Im}(k_-)|$, must be positive. In other words, the spatial growth rate, $|\text{Im}(k_+)|$, which lies above the separatrix, must surpass the spatial damping rate, $|\text{Im}(k_-)|$.

![Figure 6: Schematic drawing of the pair of roots that explain the oscillation and stopping of oscillation in a plasma BWO in (1) low, (2) intermediate, and (3) high plasma densities, respectively. The pair of roots marked by solid and open circles correspond, respectively, to the oscillation and the stopping of oscillation in the plasma BWO.](image-url)
which lies below the separatrix. The roots \( k_+ \) and \( k_- \) obtained from Eqs. (3), (4), and (5), therefore, must satisfy the following two conditions simultaneously in order to have coherent oscillation in a finite length structure: (1) positive temporal growth rate, i.e., \( \omega_t > 0 \); (2) positive global spatial growth rate, i.e., \( |\text{Im}(k_+) - \text{Im}(k_-)| > 0 \). Thus the fact that the reflection coefficient is \( 0 < R < 1 \) imposes an additional restriction for having oscillation. We believe that the following example will give an exact picture of this fact. Equations (3)–(5) are solved simultaneously for the case \( N_p = 2 \times 10^{11} \text{ cm}^{-3} \) and \( I_e = 2.3 \text{ kA} \) which are the same as those in Fig. 2, for two different sets of \( L \) and \( R \). The pair of roots \( k_+ \) obtained from Eqs. (3)–(5) for \( L = 16.7 \text{ cm}, R = 0.1 \), and \( N = -1 \), are marked by solid circles and are denoted by \( a_+ \) and \( a_- \) in Fig. 2. In this case, the root \( a_+ \) above the separatrix is located deeper than the root \( a_- \) below the separatrix, i.e., \( \text{Im}(k_+) > |\text{Im}(k_-)| \), thus the pair of the roots satisfies Eqs. (6) and (7). Moreover, the condition \( \omega_t > 0 \) is met as shown by the position of \( a_- \). The waveguide with these parameters, therefore, will oscillate. In the case of \( L = 28 \text{ cm}, R = 0.02 \), and \( N = 1 \), however, the corresponding pair of the roots are marked by open circles and are denoted by \( b_+ \) and \( b_- \) in Fig. 2. The root \( b_+ \) above the separatrix is located shallower than the root \( b_- \) below the separatrix. This fact results in \( |\text{Im}(k_+)| < |\text{Im}(k_-)| \). Equation (7), therefore, cannot be satisfied and the structure will not longer oscillate for these parameters, even though the real parts of these roots satisfy Eq. (6) and \( \omega_t > 0 \), as shown by the position of \( b_+ \). It must be noted here that the present analysis is restricted to the case of \( N = \pm 1 \) only, simply because the solutions are easily found. We cannot find oscillation condition for \( N = 0 \), because, in this case, \( \text{Re}(k_+) = \text{Re}(k_-) \) in Eq. (6), thus \( k_+ \) cannot locate below \( k_- \), as is inspected in Fig. 2. However, there may exist possibilities of oscillation for \( |N| > 1 \).

We have found that the operation of the plasma-filled BWO can be divided into three major cases depending on the filled plasma density, namely, the low, intermediate, and high \( N_p \) regions. These are shown schematically in Fig. 6. In vacuum or low \( N_p \) case (1), the separatrix is a curve extending from left bottom to right top. For high \( N_p \) case (3), the separatrix is from left top to right bottom; and in intermediate values of \( N_p \) in case (2), the separatrix is a horizontal curve. For given values of \( L \) and \( R \), the pairs of \( k_+ \) and \( k_- \) satisfying mathematically Eqs. (3)–(5) are connected to each other by solid lines (\( N = -1 \)) and dashed lines (\( N = 1 \)), respectively, in Fig. 6. In (1) low \( N_p \) case, the pair of roots for \( N = -1 \), shown by the solid circles and defined here as the \( S \) type of pair, are only found to cause coherent oscillation, as is expected from Eqs. (6) and (7). However, with the increase in \( N_p \), the separatrix gets horizontal and the \( S \) type of pair of roots for \( N = -1 \) changes to \( T \) type of pair as shown for (2) intermediate \( N_p \) case in Fig. 6. Both pairs of roots for \( N = -1 \) \((T \text{ type})\) and \( N = 1 \) \((U \text{ type})\) are found not to satisfy the conditions for coherent oscillation required by Eqs. (6) and (7) for such \( N_p \)'s. Further increase in \( N_p \), however, is found to change the \( U \) type of pair of roots in the intermediate \( N_p \) into \( V \) type of pair of roots for \( N = 1 \) in the case (3) of high \( N_p \). The \( V \) type of pair of roots shown by the solid circles for \( N = 1 \) in the high \( N_p \) case (3) are found to cause again coherent oscillation since they satisfy Eqs. (6) and (7). The above discussion reveals that the output radiation from a plasma BWO may stop for an intermediate range of plasma densities.

The growth rates in \( L = 16.7 \text{ cm} \) long plasma-filled corrugated waveguide are shown in Fig. 7. The beam current and the reflection coefficient, \( R \), were held constant at 2.3 kA and 0.1, respectively. The solid and dashed lines are, respectively, for \( N = -1 \) and \( N = 1 \). The dot-dashed lines show the growth rates for \( L \) infinity. Regions (1) and (3) denoted by OSC (thick solid curves) and those (2), (4), and (5) denoted by STOP correspond, respectively, to the oscillation and stopping oscillation as functions of the plasma density \( N_p \).
The oscillation of TM is avoided in our formulation, because Eq. (7) becomes infinite. Nevertheless, we have found that our starting currents for $N_p = 0$ are not contradictory to the results by Swegle and by others. Details will be published elsewhere. The main difference is as follows; they sometimes chose $|N| > 1$, whereas we have chosen $N = \pm 1$ in the present paper.

In Fig. 9, the oscillation regions are hatched in a parameter plane of the reflection coefficient, $R$, versus the waveguide length $L$. The beam current is $I_b = 2.3$ kA. The solid and dashed lines correspond to the plasma density $N_p = 2 \times 10^{11}$ and $1.5 \times 10^{12} \text{ cm}^{-3}$, respectively. The thinner parts of the curves are due to $\omega_p = 0$. The thicker parts of the curves, on the other hand, come from a negative global spatial growth rate. The starting length for oscillation (thinner curves) increases roughly as the reflection coefficient, $R$, decreases. We observed that the starting length of waveguide filled with plasma for which $N = -1$ (case of $N_p = 2 \times 10^{11} \text{ cm}^{-3}$) is less than that of a vacuum waveguide. The starting length at $N_p$’s for which $N = 1$ (case of $N_p = 1.5 \times 10^{12} \text{ cm}^{-3}$) is, however, higher than that of the vacuum case. Over a certain range of the reflection coefficient, $R$, the waveguide is again found to stop oscillation within an intermediate range of waveguide lengths. The points marked by A and B in this figure correspond, respectively, to the cases for which the roots $k$ obtained from Eqs. (3)–(5) are shown in Fig. 2 by the notations $a_{\pm}$ and $b_{\pm}$.

Finally, Fig. 10 shows the oscillation region by hatching in a parameter plane of reflection coefficient, $R$, versus the plasma density, $N_p$, computed under the assumption that $I_b$
$= 2.3 \text{kA and } L = 16.7 \text{ cm.}$ The temporal growth rate, $\dot{\omega}_I$, becomes negative when the waveguide is overdriven with plasma above the density determined by the thinner curve, in which case the system becomes plasma TWT. The thicker part of the curve arises from the failure in satisfying the condition of positive global spatial growth rate. Here again, an intermediate range of plasma densities is found to stop oscillation for $R < 0.98$.

### IV. DISCUSSION AND CONCLUSIONS

A linear theory for the small signal gain of a plasma-filled backward wave oscillator, driven by an intense relativistic electron beam guided by an infinitely strong axial magnetic field, has been presented assuming a localized source of perturbation. For such a source, the existence of the saddle point with $\omega_o > 0$ determines the instability of the infinitely long system. As is well-known, the location of the saddle point can be found by solving the simultaneous equations $D(k,\omega) = 0$ and $\partial D(k,\omega)/\partial k = 0$. Instead of solving these equations, we have devised a convenient numerical technique in the present work to find the locations of the saddle point with sufficient accuracies as much as we want. It is to calculate the finite length waveguide case with exceedingly large $L$. For $L$ infinity, Eq. (3) yields $k_+ = k_-$, which is nothing but the saddle point. This fact would help us to determine the continuous change in the position of the saddle point over a wide range of plasma density. The results in Figs. 4 and 5 are obtained from Eqs. (3)–(5) with an exceedingly large value of $L$ (8500 cm) and $R = 0.999$ by using the Newton iteration method.\textsuperscript{12}

The linear analysis predicts a considerable increase both in the spatial growth rate and in the oscillation frequency with increase in $N_p$. In general, the relative merits of the spatial and temporal growth rates as a measure of the strength of the instability at linear stage depend on the waveguide length as well as on other experimental conditions. In general, both of the linear growth rates have nothing to do with the radiation output. For example, if the structure length and the time of the development of the instability are both sufficiently large, the output radiation will be deter-

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**FIG. 10.** Hatched oscillation region in the parameter plane of plasma density, $N_p$, and reflection coefficient, $R$. $I_e = 2.3 \text{kA and } L = 16.7 \text{ cm.}$

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**FIG. 11.** Explanatory drawing for possible choices of mode conversion due to end reflections: (a) Schematic view of the roots on the dispersion curve: (i) two-wave interaction, (ii) three-wave interaction. (b) A possible model for three-wave interaction.
I must be small for a given \( I \), and \( k_2 - a_+ \), \( + I \), \( \omega I \) near \( a' + \), \( a_+ \), \( \eta \), and \( e \) are assumed to be related to each other by the reflection coefficients \( S_1 \) and \( S_2 \), respectively, at \( z = 0 \) and \( z = L \). Similarly, \( a_+ \) and \( a'_- \) are assumed to be related to each other by the reflection coefficients \( S_3 \) and \( S_4 \), respectively, at \( z = 0 \) and \( z = L \). The angular frequency \( \omega \) and the wave numbers \( a_+ \), \( a_- \), and \( a'_- \) corresponding to the oscillation of the finite length system can be determined by solving the following equations simultaneously:

\[
\begin{align*}
D(a_+, \omega) &= 0, \\
D(a_+ + \epsilon, \omega) &= 0, \\
D(a_+ + \epsilon - \eta, \omega) &= 0, \\
R \exp(-ieL) + S \exp(-inL) &= 1,
\end{align*}
\]

where \( e = a_+ - a_- \), \( \eta = a_+ - a'_- \), \( R = |R_1 R_2| \), and \( S = |S_1 S_2| \). Assuming \( R = S = 0.1 \) and \( L = 16.7 \) cm, we have solved the above equations for various plasma densities to determine the starting beam current for oscillation and the results are shown by the solid triangles in Fig. 8. For a large \( N_p \) \((N = 1)\), the BWO may recover again, because the interaction approaches \( 2\pi \)-mode. In the case of nonzero \( I_b \), also, similar situation to the case of \( I_b = 0 \) may happen. Increase in the group velocity is estimated as follows. The intermission of the BWO occurred, when the separatrix becomes a horizontal curve, as was shown in case (2) in Fig. 6. The group velocity \( v_g = |\partial \omega / \partial k_c| = |\partial \omega / \partial k_b| = |\Delta \omega / \Delta k_b| \) as given by Eq. (2.6.8) in Ref. 25 or equivalently by Cauchy–Riemann relationship for a complex function \( \omega = f(k) \). When the separatrix near the saddle point in complex \( k \) plane is horizontal, \( |\Delta k_b| \) must be small for a given \( \Delta \omega_c \) i.e., an amount of approach to real axis in complex \( \omega \) plane keeping \( \omega_c \) constant. This is because the solid arrow lines in Fig. 2 must be parallel to the separatrix near the saddle point. Accordingly, \( v_g \) is large, and beam–structure wave interaction becomes weak. We cannot find the pair of \( k_+ \) and \( k_- \) for coherent oscillation, as was shown in case (2) in Fig. 6, if \( L \) or \( R \) is small. On the other hand, for low \( N_c \) or high \( N_p \) case, \( v_g \) is small because of increased \( |\Delta k_b| \) and we may have a possibility to find appropriate pair of \( k_+ \) and \( k_- \) for coherent oscillation which satisfy Eqs. (3)–(7), even if \( L \) or \( R \) is small.

In conclusion, in finite length plasma-filled BWO's, we have found that there are two reasons that can prevent \( N = \pm 1 \) oscillation; (1) the negative temporal growth rates, and (2) the negative global spatial growth rates. The second condition comes from the requirement that the spa-
tial growth rate of the beam space-charge wave must be greater than the spatial damping rate of the backward structure wave for effective positive feedback, because the waveguide is leaky, i.e., $0 < R < 1$. As a result of the second condition, coherent oscillation in the plasma-filled BWO's stops for intermediate ranges of plasma density, beam current, and even for structure length.

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22 Details of the generalized schemes will be published elsewhere.