STRONGLY GRADED HEREDITARY ORDERS

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This paper is dedicated to the memory of Professor Dennis Estes.

Abstract. Let R be a Dedekind domain with global quotient field K. The purpose of this note is to provide a characterization of when a strongly graded R-order with semiprime 1-component is hereditary. This generalizes earlier work by the first author and G. Janusz in (J. Haefner and G. Janusz, *Hereditary crossed products*, Trans. Amer. Math. Soc. **352** (2000), 3381-3410).

Recall that, for a Dedekind domain R with quotient field K, an R-*order* in a separable K-algebra A is a module-finite R-algebra Λ, contained in A, such that $K\Lambda = A$. For a group G, we say that the R-order Λ is *strongly G-graded* provided there is a decomposition $\Lambda = \oplus_{g\in G}\Lambda_g$ with $\Lambda_g\Lambda_h = \Lambda_{gh}$ for all $g, h \in G$. If 1 denotes the identity element of G, then Λ_1 is a subring of Λ, which we denote by Δ. We write $\Lambda = \Delta(G)$ to indicate that Λ is strongly G-graded with identity component Δ. In this note, we consider the following problem:

> **The hereditary problem for strongly graded orders:** Determine necessary and sufficient conditions on G, the grading imposed by G, and Δ to ensure that Λ is hereditary.

Our general solution to this problem appears in Theorem 7. The idea of the proof is to use Morita theory to reduce to the case where Λ is a crossed product and then apply a result of [4], which we describe next.

Recall that a strongly G-graded R-order $\Lambda = \Delta(G)$ is a *crossed product order* provided for each $g \in G$, $\Lambda_g \cong \Delta$ as left Δ-modules. In this case, there exist $u_g \in \Lambda^*$ (the unit group of Λ) such that $\Lambda_g = \Delta u_g$ for all $g \in G$. Moreover, there exist a group homomorphism $\alpha : G \to \mathrm{Aut}_R(\Delta)$ (the "action of G on Δ") and a cocycle $\tau \in Z^2(G, R^*)$ (the "twisting of the action of G") such that the multiplication in Λ is given by $u_g\delta = \alpha(g)(\delta)u_g$ for $\delta \in \Delta$, and $u_gu_h = \tau(g, h)u_{gh}$. (See [6] for more details on this construction.)

If $\Delta(G)$ is a crossed product order with action α, we say that a subgroup H of G *acts as central outer automorphims of* Δ provided $\alpha(H)\cap \mathrm{Inn}(\Delta) = 1$. The main result of [4] is that, if $\Lambda = \Delta(G)$ is a crossed product order, then Λ is hereditary if and only if Δ is hereditary and, for each maximal ideal $\mathfrak{m}$ of R containing a prime divisor p of $|G|$, any p-Sylow subgroup of G acts as central outer automorphisms of $\hat{\Delta}_{\mathfrak{m}}$.

Definition 1. Given a strongly G-graded ring $\Lambda = \Delta(G)$ and $g \in G$, we say g is *inner on* Δ provided $\Lambda_g \cong \Delta$ as Δ-bimodules. Otherwise, g is *outer on* Δ. For a

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subgroup H of G, set

$$\mathrm{Inn}_\Delta(H) = \{h \in H : h \text{ is inner on } \Delta\}$$

We say that H is *inner on* Δ or *inner grades* Δ if $\mathrm{Inn}_\Delta(H) = H$ and it is *outer on* Δ or *outer grades* Δ provided $\mathrm{Inn}_\Delta(H) = 1$.

We remark that the above definitions do indeed generalize the classical notion of inner actions. To see this, suppose $\Lambda = \Delta(G)$ is a crossed product order with group action $\alpha : G \to \mathrm{Aut}_R(\Delta)$ such that $\alpha(g) \in \mathrm{Inn}_R(\Delta)$ for some $g \in G$. Then it is easy to see that $\Lambda_g = \Delta u_g \cong \Delta$ as bimodules. Hence, if g acts as an inner automorphism on Δ in the classical sense, it is inner on Δ in the sense defined above.

Lemma 2. *Let $\Lambda = \Delta(G)$ be strongly graded by G. Then, for any subgroup H of G, $\mathrm{Inn}_\Delta(g^{-1}Hg) = g^{-1}\,\mathrm{Inn}_\Delta(H)g$.*

Proof. Suppose that $h \in \mathrm{Inn}_\Delta(H)$. Then $\Delta_h \cong \Delta_1$ as bimodules. It follows that $\Delta_{g^{-1}}\Delta_h\Delta_g \cong \Delta_1$ as bimodules as well. This shows that $g^{-1}\,\mathrm{Inn}_\Delta(H)g \subseteq \mathrm{Inn}_\Delta(g^{-1}Hg)$. For the converse, let $\Delta_{g^{-1}hg} \cong \Delta_1$ as bimodules. Since Δ is strongly graded, $\Delta_{g^{-1}hg} = \Delta_{g^{-1}}\Delta_h\Delta_g$. It follows that $\Delta_h \cong \Delta_1$ as bimodules, proving the reverse inclusion. □

Lemma 3. *Suppose that R is a complete DVR, and that $\Lambda = \Delta(G)$ is a strongly-graded R-order such that Δ is prime and basic. Then $\Delta(G)$ is a crossed product order.*

Proof. Since R is a complete DVR, Δ is semiperfect. Consequently there exist only finitely many indecomposable projective left Δ-modules up to isomorphism. Let $\{P_1, \dots, P_t\}$ be representatives for these isomorphism classes. Then there exist positive integers m_i such that $\Delta \cong \oplus_{i=1}^t P_i^{(m_i)}$ as left Δ-modules.

For $g \in G$, we have that $\Lambda_g \in \mathrm{Pic}(\Delta)$, because Λ is strongly graded. This implies that Λ_g, viewed as a left Δ-module, is isomorphic to a direct sum of the P_i. So, we may write $\Lambda_g \cong \oplus_{i=1}^t P_i^{(n_i)}$, where the n_i are *a priori* nonnegative integers. Now, since Λ_g is a progenerator and $\mathrm{End}(\Lambda_g) \cong \Delta$ as rings, a combinatorial argument shows that in fact $m_i = n_i$ for all i, so that $\Lambda_g \cong \Delta$ as left Δ-modules. Thus Λ is a crossed product. □

We can now state and prove the prime case of our solution to the hereditary problem. We use the following notation for the remainder of the paper. R denotes a Dedekind domain whose quotient field K is a global field and A denotes a separable K-algebra. For a maximal ideal $\mathfrak{m}$ of R, $\hat{R}_\mathfrak{m}$ denotes the completion of R at $\mathfrak{m}$. Similarly, Δ denotes an R-order in A, and $\hat{\Delta}_\mathfrak{m}$ denotes the $\hat{R}_\mathfrak{m}$-order $\hat{R}_\mathfrak{m} \otimes_R \Delta$.

Theorem 4. *Assume that A is simple (so Δ is prime). Then Λ is hereditary if and only if Δ is hereditary and, for each maximal ideal $\mathfrak{m}$ of R containing a prime divisor p of $|G|$, some (hence every) p-Sylow subgroup of G is outer on $\hat{\Delta}_\mathfrak{m}$.*

Proof. First note that the induction functor $\Lambda \otimes_\Delta - : \Delta - \mathrm{mod} \to \Lambda - \mathrm{mod}$ is separable [4, Proposition 2.2]. It follows by [4, Proposition 2.3] that if Λ is hereditary, then so is Δ. Thus, for the remainder of the proof, we assume that Δ is hereditary. Next observe that any two p-Sylow subgroups of G are conjugate. Thus, by Lemma 2, it suffices to verify that $\mathrm{Inn}_{\hat{\Delta}_\mathfrak{m}}(P) = 1$ for a single p-Sylow subgroup P. By [7, Theorem 40.5], Λ is hereditary if and only if $\hat{\Lambda}_{\mathfrak{m}}$ is hereditary for each maximal ideal $\mathfrak{m}$. We show that Λ is hereditary if and only if $\hat{\Lambda}_{\mathfrak{m}}$ is hereditary for those maximal ideals $\mathfrak{m}$ containing a prime divisor of $|G|$.

To see this, assume $\hat{\Lambda}_{\mathfrak{m}}$ is hereditary for those maximal ideals $\mathfrak{m}$ containing a prime divisor of $|G|$. Fix an arbitrary maximal ideal $\mathfrak{m}$ of R. If it contains no prime divisors of $|G|$, then $|G|$ is a unit in $\hat{R}_{\mathfrak{m}}$. By Proposition 2.2 of [4], we have that $\hat{\Lambda}_{\mathfrak{m}}$ is a separable extension of $\hat{\Delta}_{\mathfrak{m}}$. Since Δ is hereditary, so is $\hat{\Delta}_{\mathfrak{m}}$ and it follows from [4, Proposition 2.3] that $\hat{\Lambda}_{\mathfrak{m}}$ is hereditary. Thus, to verify whether Λ is hereditary, it suffices to check $\hat{\Lambda}_{\mathfrak{m}}$ at those maximal ideals $\mathfrak{m}$ containing prime divisors of $|G|$.

Assume that $\mathfrak{m}$ contains a prime divisor p of $|G|$, and fix a p-Sylow subgroup P of G. For ease of notation, we write $\hat{R}$ for $\hat{R}_{\mathfrak{m}}$, etc. Since $\hat{\Delta}$ is prime hereditary, there is an idempotent e of $\hat{\Delta}$ such that $e\hat{\Delta}e$ is basic hereditary. We claim that $e\hat{\Lambda}e$ is a strongly G-graded order with components $e\hat{\Lambda}_x e$ for $x \in G$. To see this, fix homogeneous components $e\hat{\Lambda}_g e, e\hat{\Lambda}_h e$. Then,

$$\begin{aligned} e\hat{\Lambda}_g e \cdot e\hat{\Lambda}_h e &= e\hat{\Lambda}_g e\hat{\Lambda}_h e \\ &= e\hat{\Lambda}_g \hat{\Lambda}_1 e \hat{\Lambda}_1 \hat{\Lambda}_h e \quad (\text{since } \hat{\Lambda} \text{ is strongly } G\text{-graded}) \\ &= e\hat{\Lambda}_g \hat{\Delta} \hat{\Lambda}_h e \quad (\text{since } \hat{\Lambda}_1 = \hat{\Delta} \text{ and } \hat{\Delta} e \hat{\Delta} = \hat{\Delta}) \\ &= e\hat{\Lambda}_{gh} e \quad (\text{since } \hat{\Lambda} \text{ is strongly } G\text{-graded}) \end{aligned}$$

which shows that $e\hat{\Lambda}e$ is strongly graded.

The orders $\hat{\Lambda}$ and $e\hat{\Lambda}e$ are Morita equivalent via the pair of graded progenerators $\hat{\Lambda}e$ and $e\hat{\Lambda}$. In addition, this equivalence preserves, in a certain sense, the grading of the two orders. This is an example of what is called a *graded equivalence*; see [2] for more information on graded equivalences.

By Morita equivalence, $\hat{\Lambda}$ is hereditary if and only if $e\hat{\Lambda}e$ is hereditary. Now, the identity component of $e\hat{\Lambda}e$ is $e\hat{\Delta}e$, which is basic, prime and hereditary. Since $e\hat{\Lambda}e$ is strongly G-graded, we see by Lemma 3 that $e\hat{\Lambda}e$ is a crossed product order. Hence, we may apply [4, Theorem 6.8] to conclude that $e\hat{\Lambda}e$ is hereditary if and only if P acts as central outer automorphisms. As we have remarked above, this is equivalent to saying that P outer grades $e\hat{\Lambda}e$. To finish the proof, we note that P outer grades $e\hat{\Lambda}e$ if and only if P outer grades $\hat{\Lambda}$, because $\hat{\Lambda}_x \cong \hat{\Delta}$ as bimodules if and only if $e\hat{\Lambda}_x e \cong e\hat{\Delta}e$ for any $x \in P$. (This uses the fact that $\hat{\Lambda}e$ and $e\hat{\Lambda}$ induce a graded equivalence between $\hat{\Lambda}$ and $e\hat{\Lambda}e$.) Thus, $\hat{\Lambda}$ is hereditary if and only if $\mathrm{Inn}_{\hat{\Lambda}}(P) = 1$. □

The above proof requires us to reduce to the case when Δ is basic so that Λ is a crossed product order. We present an example that shows that, even over a complete DVR, a strongly graded order with non-basic hereditary 1-component need not be a crossed product order. Thus Lemma 3 is the best possible, and the proof of Theorem 4 cannot be simplified in this regard. The example depends on the following basic construction technique for strongly graded rings, which is a special case of [5, p. 23].

Construction. Let Δ be a ring, and let $X \in \mathrm{Pic}(\Delta)$ have finite order n. Let G denote the cyclic group of order n with generator g. We construct a G-strongly graded ring Λ with 1-component Δ as follows: Set $\Lambda = \oplus_{i=1}^{n} \Lambda_{g^i}$, where $\Lambda_{g^i} = X^i$, and define the multiplication by the tensor product. In other words, fix isomorphisms

$X^i \otimes X^j \cong X^{i+j}$ which are compatible in the obvious sense. Then, given homogeneous elements x_i, x_j in Λ_{g^i}, Λ_{g^j}, respectively, we define $x_i \cdot x_j = x_i \otimes x_j \in X^{i+j}$ (using the fixed isomorphism). Note that $\Lambda_1 = \Lambda_{g^0} = X^0 \cong \Delta$, so the 1-component is Δ, as claimed. Note also the grading is strong, as $X^i \otimes X^j \cong X^{i+j}$ for all i, j by construction.

Example 5 (Strongly graded orders need not be crossed products)**.** Let Δ be a non-basic hereditary order over a complete DVR, and let Γ be the associated basic order $e\Delta e$. Now, $\mathrm{Picent}(\Delta) \cong \mathrm{Picent}(\Gamma)$ is cyclic (say of order n) generated by $\mathrm{rad}(\Delta)$ (respectively $\mathrm{rad}(\Gamma)$). The fact that it is cyclic is in [4], and the fact that it is generated by the radical follows from [7], where it is shown that $\mathrm{rad}(\Delta)$ has the correct order. Now, if Δ is not basic, then $\mathrm{rad}(\Delta)^k$ is not principal as a left ideal for any $1 \leq k < n$, and so $\mathrm{rad}(\Delta)^k \ncong \Delta$ as bimodules for any $1 \leq k < n$. Thus, the $\mathbb{Z}/n\mathbb{Z}$-strongly graded order $\Delta(G) = \oplus_{k=0}^{n-1} \mathrm{rad}(\Delta)^k$ is not a crossed product order, even though $e\Delta(G)e$ is. □

Having dealt with the prime case, we turn our attention to the semiprime case. To begin, we investigate the action of G that is imposed on the central idempotents of Δ. We fix the following notation.

Notation. Given $\Lambda = \Delta(G)$, suppose that $\Delta = \Delta_1 \oplus \cdots \oplus \Delta_t$ is a direct sum of prime rings. Let $e_1, \ldots, e_t$ denote the orthogonal central idempotents of Δ. Then the group G acts on the e_i, by $e_i \Lambda_g = \Lambda_g e_{g(i)}$. (This is a special case of the action of $\mathrm{Pic}(\Delta)$ on the center of Δ; see [1, §55].) Suppose that this action partitions $\{e_i\}$ into m orbits, and let $\varepsilon_1, \ldots, \varepsilon_m$ be a set of representatives for the equivalence classes under this action. Finally, let G_i denote the stabilizer of ε_i.

Lemma 6. *Assume the above notation.*

(a) Λ *is Morita equivalent to* $\oplus_{i=1}^m \varepsilon_i \Lambda \varepsilon_i$.
(b) *Each* $\varepsilon_i \Lambda \varepsilon_i$ *is strongly graded by* G_i*, with identity component* $\varepsilon_i \Delta \varepsilon_i$*, a prime ring.*

Proof. This is proven in [3, Theorem 5.4], under the assumption that Δ is maximal. However, examining the proof, we see that the above is true without this hypothesis. □

Theorem 7. *Let R be a Dedekind domain with global quotient field K. Let A be a semisimple K-algebra, and Δ be an R-order in A. (Note that Δ is necessarily semiprime.) Suppose that $\Lambda = \Delta(G)$ is a strongly graded R-order. Then $\Delta(G)$ is hereditary if and only if Δ is hereditary, and, in the notation of Lemma 6, the following conditions hold, for $1 \leq i \leq m$.*

> *For each maximal ideal $\mathfrak{m}$ of R containing a prime divisor p of $|G_i|$, $\mathrm{Inn}_{\varepsilon_i \hat{\Delta} \varepsilon_i}(P) = 1$ for some (hence every) p-Sylow subgroup P of G_i.*

Proof. Note that, as in the proof of Theorem 4, we conclude that Δ being hereditary is a necessary condition. Thus, we may assume Δ is hereditary, and then Δ decomposes as a direct sum of prime rings. Hence, Lemma 6 applies.

Since Λ is Morita equivalent to $\oplus_{i=1}^m \varepsilon_i \Lambda \varepsilon_i$, it follows that Λ is hereditary if and only if each $\varepsilon_i \Lambda \varepsilon_i$ is. Now, each $\varepsilon_i \Lambda \varepsilon_i$ has prime identity component, so that Theorem 4 applies. □

We close this paper with some remarks and examples. First, the decomposition $\oplus_i \varepsilon_i \Lambda \varepsilon_i$ depends upon the choice of the representatives ε_i of the orbits of $\{e_1, \ldots, e_t\}$. If we chose different representatives ε_i', then the new stabilizer groups G_i' would be conjugate to the original groups G_i. If we choose p-Sylow subgroups P, P' of G_i, G_i' respectively, then $\mathrm{Inn}_{\varepsilon_i \hat{\Delta} \varepsilon_i}(P) = 1$ if and only if $\mathrm{Inn}_{\varepsilon_i' \hat{\Delta} \varepsilon_i'}(P') = 1$, by Lemma 2. Thus, the choice of representatives does not affect the application of Theorem 7.

Second, the statement of Theorem 7 requires checking whether or not $\mathrm{Inn}_{\hat{\Delta}}(P) = 1$ at various completions. It is not enough to assume that $\mathrm{Inn}_{\Delta}(P) = 1$, i.e. that P outer grades Δ globally, because the property of being outer is not a local-global property. The next example illustrates this fact.

Example 8 (Outer grading is not a local-global property)**.** Let R be the ring of Gaussian integers $\mathbb{Z}[i]$, and let K denote the quotient field $\mathbb{Q}(i)$. The prime integer 5 is contained in exactly two ideals of R: $\mathfrak{p} = (1+2i)$ and $\mathfrak{q} = (1-2i)$. Let I denote the ideal (5), and let Δ denote the tiled order

$$\Delta = \begin{pmatrix} R & R & R & R & R \\ I & R & R & R & R \\ I & I & R & R & R \\ I & I & I & R & R \\ I & I & I & I & R \end{pmatrix}.$$

We first compute $\mathrm{Picent}(\Delta)$. By Fröhlich's Theorem [1, Theorem 55.25] , there is an isomorphism

$$\mathrm{Picent}(\Delta) \cong \bigoplus_{\mathfrak{m}\text{ maximal}} \mathrm{Picent}(\hat{\Delta}_{\mathfrak{m}}).$$

(Here we are using that R is a PID.) Note that, if $\mathfrak{m} \neq \mathfrak{p}, \mathfrak{q}$, then $\hat{I}_{\mathfrak{m}} \cong \hat{R}_{\mathfrak{m}}$, and so $\hat{\Delta}_{\mathfrak{m}} = M_5(\hat{R}_{\mathfrak{m}})$. In particular, $\mathrm{Picent}(\hat{\Delta}_{\mathfrak{m}}) = 0$ if $\mathfrak{m} \neq \mathfrak{p}, \mathfrak{q}$.

If $\mathfrak{m} = \mathfrak{p}$, then $\hat{\Delta}_{\mathfrak{p}}$ is the unique basic hereditary order in $M_5(\hat{K})$, and so $\mathrm{Picent}(\hat{\Delta}_{\mathfrak{p}}) \cong \mathbb{Z}/5\mathbb{Z}$, by [7, Theorem 39.18]. Similarly, $\mathrm{Picent}(\hat{\Delta}_{\mathfrak{q}}) \cong \mathbb{Z}/5\mathbb{Z}$. Thus, $\mathrm{Picent}(\Delta) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.

Now, let X be the bimodule generating the subgroup $0 \oplus \mathbb{Z}/5\mathbb{Z}$ of $\mathrm{Picent}(\Delta)$ (i.e. the component corresponding to $\hat{\Delta}_{\mathfrak{q}}$), and form the strongly $\mathbb{Z}/5\mathbb{Z}$-graded order Λ, where $\Lambda_{g^i} = X^i$. Note that, globally, Λ is outer graded (because $X^i \cong \Delta$ as bimodules if and only if $i = 0$). However, if we pass to the completion at $\mathfrak{p}$, then $\hat{X}_{\mathfrak{p}} \cong \hat{\Delta}_{\mathfrak{p}}$ as bimodules, by construction. Thus, it is possible for a global outer grading to become inner at the completion. □

Finally, Theorem 7 requires checking the condition on the grading at each $\varepsilon_i \Lambda \varepsilon_i$ (in the notation of Lemma 6), rather than simply checking the grading on Λ. That is, it is not enough to verify that $\mathrm{Inn}_{\hat{\Lambda}}(P) = 1$ for p-Sylow subgroups of each G_i (or of G). This is because the Morita equivalence between Λ and $\oplus_{i=1}^m \varepsilon_i \Lambda \varepsilon_i$ is not a graded equivalence. Hence, the property of being an outer grading is not preserved under this correspondence. Our last example illustrates this.

Example 9 (Passing to $\varepsilon_i \Lambda \varepsilon_i$ can change the grading)**.** Let Δ be a prime, hereditary order, and let $\Delta^{(d)}$ denote the direct sum of d copies of Δ. The symmetric group S_d acts on $\Delta^{(d)}$ by permuting coordinates, so that we may form the crossed product (with trivial twisting) $\Delta^{(d)}(S_d)$ relative to this action. In the notation of

Lemma 6, $G = S_d$ acts transitively on $\{e_1, \dots, e_d\}$. Let us fix $\varepsilon_1 = e_1$, so that $G_1 = S_{d-1}$, embedded in S_d as those permutations that fix the first coordinate.

It is straightforward to compute that $e_1 \Delta^{(d)}(S_d) e_1 \cong \Delta S_{d-1}$, the ordinary group ring of S_{d-1} over Δ. Note that, for *any* p-Sylow subgroup P of S_{d-1}, $\mathrm{Inn}_{\Delta S_{d-1}}(P) = P$. However, if we view P as a subgroup of S_d, then $\mathrm{Inn}_{\Delta^{(d)}(S_d)}(P) = 1$. This is because, for $\pi \in P$, $\Delta^{(d)}(S_d)_\pi \not\cong \Delta^{(d)}$ as bimodules. (The action on the right is twisted by π, and π is not an inner automorphism of Δ.) Thus, it is necessary when applying Theorem 7 to consider the grading on each $\varepsilon_i \Lambda \varepsilon_i$, and not simply the grading on Λ. □

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